

## THE CRITICAL SET OF A DEMAND CORRESPONDENCE IN THE PRICE SPACE AND THE WEAK AXIOM OF REVEALED PREFERENCE†

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### I. Introduction

The continuity property of demand is one of basic requirements in the general equilibrium analysis of market economies. For this reason the "classical" general equilibrium theory [see e.g. Debreu (1959) or Nikaido (1968)] confined itself to convex environments of economies so that induced demands do satisfy a basic continuity property in general. In an extended framework of the classical model where preference relations may be non-transitive, incomplete, or nonconvex, and/or where consumption sets may not be convex [see e.g. Sonnenschein (1971), Mas-Colell (1974, 1977), and Yamazaki (1978)], possible "discontinuities" of consumers' demand behavior become conspicuous. Here we would like to distinguish two types of "discontinuities." One refers to the lack of upper hemi-continuity of a demand correspondence which is called *the discontinuity of the first kind*. The other refers to the nonsingle-valuedness of the correspondence, and is called *the discontinuity of the second kind*. [The use of the term "discontinuity" in the latter sense may be justified by the facts that (1) an upper hemi-continuous correspondence need not possess a continuous selection from it and (2) even if it admits, locally or globally, a continuous selection, the market behavior of consumer need not warrant locally continuous demand since his market behavior may not coincide with the one implied by the selection.]

Corresponding to these two types of discontinuities one could consider two "critical sets" in the price and wealth space. It is known that it is most convenient to consider the two sets in separate spaces: one in the wealth space and the other in the price space. Given a price vector, *the critical set of a demand correspondence in the wealth space* is the set of all wealth levels at which the demand correspondence could have the discontinuity of the first kind. It was found that the critical set in the wealth space is at most countable [Yamazaki (1978)]. One could say that this basic property of the critical set in the wealth space is essentially the property of the consumption set which underlies the given demand correspondence, in the sense that the critical set can be characterized by the property of the consumption set and the given price vector.

On the other hand, given a wealth level *the critical set of a demand correspondence in the price space* will be defined below (in Section 2) as a generalization of the idea of the set where the demand correspondence has the discontinuity of the second kind. It was proved by Mas-Colell and Neuefeind (1977) that this critical set is of Lebesgue measure zero in the

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price space provided that the underlying preference relation of the demand correspondence is a complete, continuous, and locally nonsatiated preorder (where the consumption set may be nonconvex).

The present note is concerned with this fundamental property of the critical set in the price space. Its purpose is twofold. The first is to show that the fundamental property can be regarded as a basic property of demand rather than that of a preference relation on the consumption set generating it. In other words, we shall prove, uniquely on the basis of the (generalized) weak axiom of revealed preference, that the critical set of a demand relation in the price space has Lebesgue measure zero. The second is to point out that a dual approach using the dual preference relation on the price space induced by the demand correspondence is available and that the "critical behavior" of demand produces the corresponding "critical behavior" of the dual preference relation.

The paper is organized as follows: Section 2 defines a demand correspondence and its critical set in the price space. The statement of the main theorem is given there. Section 3 gives a proof of the theorem using the "primal" approach. In Section 4 we present a dual proof using the dual preference relation defined on the price space.

## II. Demand Correspondence and Its Critical Set in the Price Space

Let  $l$  be a positive integer which represents the number of commodities.  $R^l_+$  is the commodity space (but perfect divisibility is not assumed). A consumer is characterized by his demand for  $l$  commodities which is a correspondence from the price and the wealth space into the commodity space.  $P$  will denote the set of all strictly positive price vectors.

For convenience we shall use the following notation which describe subsets of  $R^l$ . Let  $p (\neq 0) \in R^l$  and  $b \in ]-\infty, \infty[$ .

$$\begin{aligned} H(p, b) &:= \{x \in R^l \mid p \cdot x = b\}, \\ H_{\leq}(p, b) &:= \{x \in R^l \mid p \cdot x \leq b\}, \\ H_{<}(p, b) &:= \{x \in R^l \mid p \cdot x < b\}. \end{aligned}$$

$H_{\geq}(p, b)$  and  $H_{>}(p, b)$  are defined similarly.  $H(p, b)$  is a hyperplane, and  $H_{\leq}(p, b)$ ,  $H_{<}(p, b)$ , etc. are closed or open halfspaces in  $R^l$ .

**Definition:** A (nonempty-valued) correspondence  $d$  from  $P \times ]0, \infty[$  into  $R^l_+$ , having a measurable graph, is called an (individual) demand correspondence if it has the following properties:

(1) *Budget identity:*

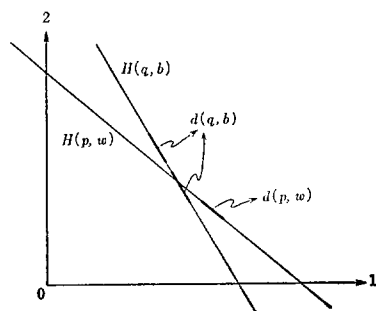
for every  $(p, w)$  in  $P \times ]0, \infty[$ ,  $d(p, w) \subset H(p, w)$ .

(2) *Generalized weak axiom of revealed preference* (abbreviated as *GWARP*):

for every pair  $(p, w)$  and  $(q, b)$  in  $P \times ]0, \infty[$ ,  $d(q, b) \cap H_{<}(p, w) \neq \emptyset$  and  $d(q, b) \cap H_{\leq}(p, w) \neq d(p, w) \cap H_{\leq}(q, b)$  imply  $d(p, w) \cap H_{\leq}(q, b) = \emptyset$ . (See Figure 1.)

**Remark 1:** Let  $d: P \times ]0, \infty[ \rightarrow R^l_+$  be a function. The weak axiom of revealed preference states as follows: for every  $(p, w)$  and  $(q, b)$  in  $P \times ]0, \infty[$ ,  $d(q, b) \in H_{\leq}(p, w)$  and  $d(q, b) \neq d(p, w)$  imply  $d(p, w) \notin H_{\leq}(q, b)$ . [Samuelson (1938)]. Kihlstrom *et al.* (1976) introduced the weak

FIGURE 1



weak axiom of revealed preference (WWARP) which states: for every  $(p, w)$  and  $(q, b)$  in  $P \times ]0, \infty[$ ,  $d(q, b) \in H_<(p, w)$  and  $d(q, b) \neq d(p, w)$  imply  $d(p, w) \notin H_<(q, b)$ . The GWARP introduced above generalizes the WWARP to incorporate demand correspondences.

For  $S \subset R^l$ ,  $\text{co } S$  denotes its convex hull.  $H_S$  is the affine subspace generated by  $S$ .  $L_S$  denotes the linear subspace parallel to  $H_S$  (i.e.,  $H_S = q + L_S$  for some  $q$  in  $R^l$ ).  $\lambda_S$  denotes the Lebesgue measure on  $L_S$ . It can naturally be considered as a measure on  $H_S$ . For  $q$  in  $R^l$ ,  $L_q$  denotes the linear subspace generated by  $q$ . If  $S$  is a convex subset of  $R^l$ ,  $\dim S$  denotes the dimension of  $L_S$ .

$d: P \times ]0, \infty[ \rightarrow R^l_+$  is a demand correspondence. Let  $w$  be a given wealth level and  $S$  a convex subset of  $P$ . Then  $d_S$  denotes the restriction of  $d$  to  $S \times \{w\}$ . We shall often write  $d$  instead of  $d_S$  when no confusion should arise from the context.  $S$  is interpreted as the set of observable price vectors. The critical set of the demand correspondence  $d$  in the price space will be defined relative to the price subspace  $S$ . By definition the critical set  $C_S$  of the demand relation  $d$  in the price subspace  $S$  is given by

$$C_S := \{p \in S \mid \dim \text{co } d(p, w) > l - \dim S\}.$$

The basic fact of the critical set  $C_S$  is that it is of Lebesgue measure zero in  $S$ . When the observable price vectors can move in every direction (i.e.,  $\dim S = l$ ), then it implies that at almost every price vector  $p$  the demand correspondence does not have the discontinuity of the second kind. In other words, it is the property of demand that the demand is almost everywhere a function.

*Theorem:*

Let  $d$  be a demand correspondence. Given a wealth level  $w$  in  $]0, \infty[$  and a convex and measurable subset  $S$  of  $P$ , the critical set  $C_S$  of the demand correspondence in the price subspace  $S$  has Lebesgue measure zero, i.e.,  $\lambda_S(C_S) = 0$ .

In the following two sections we shall present dual proofs of the main theorem.

### III. Primal Approach

Because of the property of the generalized weak axiom of revealed preference, we are

naturally interested in the behavior of halfspaces  $H_{\leq}(p, w)$  as prices vary in a particular direction. The following is an easy but a useful fact.

*Proposition 1:*

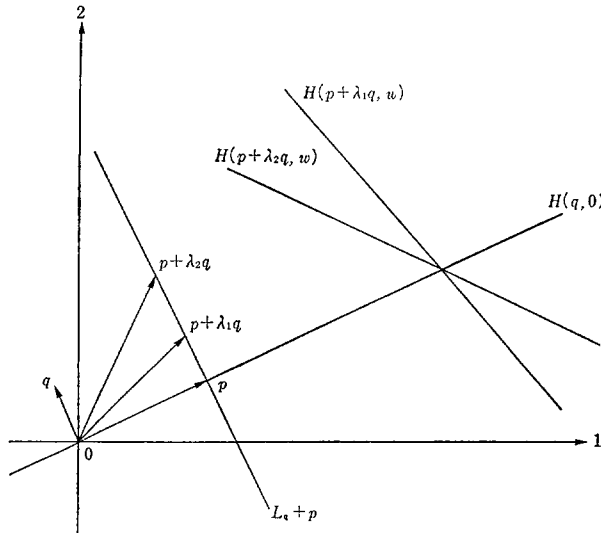
Let  $q (\neq 0) \in R^l$ ,  $p (\neq 0) \in H(q, 0)$ ,  $\lambda_2 > \lambda_1$  and  $w \in ]0, \infty[$ . Then, we have:

- (1)  $H_{\leq}(p + \lambda_2 q, w) \cap H_{>}(q, 0) \subset H_{<}(p + \lambda_1 q, w) \cap H_{>}(q, 0)$ ;
- (2)  $H_{\leq}(p + \lambda_1 q, w) \cap H_{<}(q, 0) \subset H_{<}(p + \lambda_2 q, w) \cap H_{<}(q, 0)$ .

*Proof:* (1) Let  $x \in \text{L.H.S.}$ ; then,  $(p + \lambda_2 q) \cdot x \leq w$  and  $q \cdot x > 0$ . Thus,  $p \cdot x + \lambda_1 q \cdot x < (p + \lambda_2 q) \cdot x \leq w$ . Therefore,  $x \in \text{R.H.S.}$  (2) Let  $x \in \text{L.H.S.}$  Since  $q \cdot x < 0$ , we have  $(p + \lambda_2 q) \cdot x < (p + \lambda_1 q) \cdot x \leq w$ . Thus,  $x \in \text{R.H.S.}$   $\square$

*Remark 2:* The intersection of two hyperplanes  $H(q, 0)$  and  $H(p + \lambda q, w)$  are identical for any  $\lambda \in R$ . And as shown in Figure 2, if  $\lambda_2 > \lambda_1$ ,  $H(p + \lambda_1 q, w)$  lies above  $H(p + \lambda_2 q, w)$  above  $H(q, 0)$  and the situation reverses below  $H(q, 0)$ .

FIGURE 2



*Primal Proof of the Theorem*

Define for each  $q \in L_S$ ,

$$C_q := \{p \in S \mid d(p, w) \cap H_{<}(q, 0) \neq \emptyset \text{ and } d(p, w) \cap H_{>}(q, 0) \neq \emptyset\}.$$

We first prove the following lemma:

*Lemma 2:*  $\lambda_S(C_q) = 0$  for any given  $q \in L_S$  with  $q \neq 0$ .

The measurability of  $C_q$  follows from the measurability of the graph of the demand correspondence and Proposition 4 in Hildenbrand (1974, p. 61).

By Fubini's theorem, if for every  $p \in S \cap H(q, 0)$ ,  $\# \{C_q \cap (p + L_q)\} \leq 1$ , then  $\lambda_S(C_q) = 0$ .

Assume that  $p + \lambda_2 q, p + \lambda_1 q \in C_q$  for some  $\lambda_2 > \lambda_1$ . Then, we have  $d(p + \lambda_1 q, w) \cap H_{<}(q, 0) \neq \phi$  and  $d(p + \lambda_2 q, w) \cap H_{>}(q, 0) \neq \phi$ . From Proposition 1 one obtains

- (1)  $d(p + \lambda_1 q, w) \cap H_{<}(q, 0) \subset H_{<}(p + \lambda_2 q, w) \cap H_{<}(q, 0),$
- (2)  $d(p + \lambda_2 q, w) \cap H_{>}(q, 0) \subset H_{<}(p + \lambda_1 q, w) \cap H_{>}(q, 0).$

Therefore it follows from the generalized weak axiom of revealed preference that

- (3)  $d(p + \lambda_1 q, w) \cap H_{\leq}(p + \lambda_2 q, w) = d(p + \lambda_2 q, w) \cap H_{\leq}(p + \lambda_1 q, w).$

But this is impossible. Indeed, by the budget identity  $d(p + \lambda_i q, w) \subset H(p + \lambda_i q, w)$  for  $i = 1, 2$ . But since  $H(p + \lambda_1 q, w) \cap H(p + \lambda_2 q, w) \subset H(q, 0)$ , both the L.H.S. and the R.H.S. of (3) is contained in  $H(q, 0)$ . However, (1) and (2) shows that the L.H.S. of (3) contains  $d(p + \lambda_1 q, w) \cap H_{<}(q, 0) (\neq \phi)$  whereas the R.H.S. of (3) contains  $d(p + \lambda_2 q, w) \cap H_{>}(q, 0) (\neq \phi)$ .  $\square$

Now, let  $\dim S \leq 1$ . Then, the budget identity  $\text{co } d(p, w) \subset H(p, w)$  implies that  $C_S = \phi$ , and the conclusion of the theorem is trivial. So, w.l.o.g. assume that  $\dim S \geq 2$ . Let  $L^*$  be a countable dense subset of  $L_S$ . Assume that for a price vector  $p$  in  $S$ ,  $\dim \text{co } d(p, w) > l - \dim S$ . Put  $k := \dim \text{co } d(p, w)$ . Since  $k > \dim L_S^\perp$ , there are  $(k+1)$  vectors,  $x_1, \dots, x_{k+1}$ , in  $d(p, w)$  such that  $x_{k+1} \notin L_S^\perp$  and  $k$  vectors  $x_j - x_{k+1}, j = 1, \dots, k$ , are linearly independent. Let  $\hat{x}_j$  denote the perpendicular projection of  $x_j$  on  $L_S$ . For at least one  $j$  we must have  $\hat{x}_j \neq \lambda \hat{x}_{k+1}$  for any nonzero  $\lambda$ . If not, there would exist  $\lambda_j \neq 0$  with  $\hat{x}_j = \lambda_j \hat{x}_{k+1}$  for  $j = 1, \dots, k$ . It follows that  $x_j - \lambda_j x_{k+1} \in L_S^\perp$  for each  $j = 1, \dots, k$ . Thus, there are  $\delta_j$ , not all equal to zero, such that  $\sum_{j=1}^k \delta_j (x_j - \lambda_j x_{k+1}) = 0$ . Since  $\{x_1, \dots, x_{k+1}\} \subset H(p, w)$ , we must have  $\sum_{j=1}^k \delta_j = \sum_{j=1}^k \delta_j \lambda_j$ . It follows that  $\sum_{j=1}^k \delta_j (x_j - x_{k+1}) = 0$ , contradicting the fact that  $x_j - x_{k+1}, j = 1, \dots, k$ , are linearly independent. Therefore, there exists  $j$  such that  $\hat{x}_j \neq \lambda \hat{x}_{k+1}$  for any nonzero  $\lambda$ . Thus, there exists  $q \in L^*$  such that  $\hat{x}_j \in H_{<}(q, 0)$  and  $\hat{x}_{k+1} \in H_{>}(q, 0)$ . Since  $q \cdot \hat{x}_j = q \cdot x_j$  and  $q \cdot \hat{x}_{k+1} = q \cdot x_{k+1}$ , it follows that  $x_j \in H_{<}(q, 0)$  and  $x_{k+1} \in H_{>}(q, 0)$ . This means that  $p \in C_q$ . Therefore, we have

$$C_S \subset \bigcup_{q \in L^*} C_q.$$

It then follows from Lemma 2 that  $\lambda_S(C_S) = 0$ . This completes the primal proof of the theorem.

#### IV. Dual Approach

In this section we shall present a dual proof of the main theorem. The dual approach relates the critical set of a given demand correspondence in the price space to the critical set of the dual preference relation on the price space induced by the correspondence. Thus the fundamental property of the critical set of the demand correspondence may also be understood as a basic property of the critical set of its corresponding dual preference relation.

Let  $\succsim^*$  be a binary relation on  $P$  defined as follows:

$$\begin{aligned} q \succsim^* p &\Leftrightarrow d(p, w) \cap H_{<}(q, w) = \phi \text{ or} \\ &\quad d(p, w) \cap H_{\leq}(q, w) = d(q, w) \cap H_{\leq}(p, w). \end{aligned}$$

$\succsim^*$  will be called the dual preference relation (induced by the demand correspondence at a

given wealth level  $w$ ).

*Remark 3:* Perhaps more familiar way of introducing the dual preference (to be exact, “dispreference”) relation is as follows:

- (1) Define  $\succ^*$  on  $P$  by:

$$p \succ^* q \Leftrightarrow d(p, w) \cap H_{<}(q, w) \neq \emptyset \text{ and} \\ d(p, w) \cap H_{\leq}(q, w) \neq d(q, w) \cap H_{\leq}(p, w).$$

This amounts to defining a “dual” revealed (dis-)preference relation on  $P$ . (In the context of a demand correspondence, a *dual* revealed preference seems more natural than defining a revealed preference on a consumption set.)

- (2) Define  $\succeq^*$  on  $P$  by:

$$q \succeq^* p \Leftrightarrow \text{not } [p \succ^* q].$$

Now, define a correspondence  $g^*: P \rightarrow R^l$  for  $\succeq^*$  by

$$g^*(p) := \{x \in R^l \mid q \cdot x \geq p \cdot x \text{ for all } q \succeq^* p\}.$$

The following proposition shows that  $g^*$  is nonempty-valued and has a close tie with the demand correspondence.

*Proposition 3:*

- (1)  $d(p, w) \subset g^*(p)$ .
- (2)  $g^*(p)$  is a closed convex cone.
- (3)  $\dim \text{co } d(p, w) \leq \dim g^*(p) - 1$ .

*Proof:* (1) Let  $x \in d(p, w) \setminus g^*(p)$ ; then, there exists  $q \succeq^* p$  such that  $q \cdot x < p \cdot x = w$ . Thus,  $d(p, w) \cap H_{<}(q, w) \neq \emptyset$ . Since  $q \succeq^* p$ , we must have  $d(p, w) \cap H_{\leq}(q, w) = d(q, w) \cap H_{\leq}(p, w)$ . Since  $x \notin H(q, w)$ ,  $x \notin R.H.S.$ , but  $x \in L.H.S.$  This is a contradiction. Therefore,  $d(p, w) \setminus g^*(p) = \emptyset$ , that is,  $d(p, w) \subset g^*(p)$ .

(2) Routine.

(3) It follows from (1) and (2) that we have  $\text{co } d(p, w) \subset g^*(p)$ . Let  $k := \dim \text{co } d(p, w)$ . Then, we can pick  $x_0, x_1, \dots, x_k$  from  $d(p, w)$  such that  $x_j \neq x_0$  and  $\hat{x}_j := x_j - x_0, j = 1, \dots, k$  are linearly independent. The vectors  $2x_0, x_1, \dots, x_k$  are linearly independent. Indeed, if not, there would exist  $\lambda_j, j = 0, \dots, k$ , not all zero such that  $-2\lambda_0 x_0 = \sum_{j=1}^k \lambda_j x_j$ . Since  $p \cdot x_0 = \dots = p \cdot x_k = w \neq 0$ , it follows that  $-2\lambda_0 = \sum_{j=1}^k \lambda_j$ . Therefore,  $\sum_{j=1}^k \lambda_j \hat{x}_j = 0$ , contradicting the fact that  $\hat{x}_j, j = 1, \dots, k$ , are linearly independent. Since by (1) and (2) we have  $\{2x_0, x_1, \dots, x_k\} \subset g^*(p)$ , it follows that  $\dim g^*(p) \geq k + 1$ .  $\square$

Let  $S$  be a convex subset of  $P$ . By an abuse of notation we denote the restriction of  $g^*$  to  $S$  by  $g^*$  also. No confusion should arise. The critical set of the dual preference relation  $\succeq^*$  in  $S$ , denoted by  $C_S^*$ , is defined by

$$C_S^* := \{p \in S \mid \dim g^*(p) > l - \dim S + 1\}.$$

If we put  $H^*(x, w) := \{p \in R^l \mid p \cdot x = w\}$ , then the set  $\{q \in R^l \mid q \succeq^* p\}$  is supported at  $p$  by a hyperplane  $H^*(x, p \cdot x)$  for  $x \in g^*(p)$ . If  $\dim g^*(p) = 1$ , there is a unique supporting hyper-

plane at  $p$ . If  $p$  is a critical point of the dual preference relation  $\succsim^*$ , that is, if  $p$  belongs to  $C_S^*$ , then supporting hyperplanes at  $p$  have the "degree of freedom" exceeding the difference between the "possible degree of freedom of price movements" and their "actual degree of freedom." Because of Proposition 3, the main theorem is implied by the following result concerning the critical set of the dual preference relation induced by the given demand correspondence.

**Proposition 4:**

Let  $\succsim^*$  be the dual preference relation induced by a demand correspondence  $d$  and a given wealth level  $w$ . Let  $S$  be a convex subset of  $P$ . Then, we have:

$$\lambda_S(C_S^*) = 0.$$

*Proof:* For each  $p$  in  $S$  define

$$H_p^* := [\cap_{x \in g^*(p)} H^*(x, p \cdot x)] - p.$$

$H_p^*$  is a linear subspace. Let  $\mathcal{L}_k$  denote the countable set of all  $k$ -dimensional linear subspaces of  $R^l$  with rational basis. Let  $B_n(x) \subset R^l$  be the closed ball centered at  $x$  with radius  $1/n$ . Put  $B_{n,p}(x) = B_n(x) \cap L_{g^*(p)}$ . From now on let  $k := l - \dim S + 1$ . For each  $L \in \mathcal{L}_k$ , define

$$C_{L,n}^* := \{p \in C_S^* \mid \text{there is } x (\neq 0) \in g^*(p) \text{ such that } x \in L^\perp, \\ L \cap H_p^* = \{0\}, \text{ and } B_{n,p}(x) \subset g^*(p)\}.$$

One can check that for each  $L$  and  $n$ ,  $C_{L,n}^*$  is an  $F_\sigma$ -set in  $S$ . Put  $C_L^* := \cup_n C_{L,n}^*$ . Let  $p \in C_S^*$ . Since  $\dim g^*(p) > k$ , we have  $\dim H_p^* < l - k$ . It follows that for some  $L \in \mathcal{L}_k$ , there is  $x$  with  $B_{n,p}(x) \subset g^*(p)$  for a large enough  $n$  such that  $x \in L^\perp$  and  $H_p^* \cap L = \{0\}$ . Therefore,  $C_S^* \subset \cup_{L \in \mathcal{L}_k} C_L^*$ .

It remains to show  $\lambda_S(C_{L,n}^*) = 0$ . It follows from a known property of the Hausdorff measure and its definition [see, e.g., Billingsley (1979, Section 19, pp. 208–16)] that  $\lambda_S(C_{L,n}^*)$  will be zero if we show that for any  $v \in R^l$ ,  $\#\{C_{L,n}^* \cap (v + L)\} \leq 1$  since  $\dim S = (l - k) + 1$ . Suppose that two distinct points  $p$  and  $q$  belonged to the set  $C_{L,n}^* \cap (v + L)$ . Then, by the definition of  $C_{L,n}^*$  there would exist  $x \in g^*(p)$  and  $z \in g^*(q)$  such that  $q \cdot x < p \cdot x$  and  $q \cdot z > p \cdot z$ . We shall show that these two inequalities contradict the GWARP. Indeed, if  $d(p, w) \cap H_{<}(q, w) = \emptyset$  or  $d(p, w) \cap H_{\leq}(q, w) = d(q, w) \cap H_{\leq}(p, w)$ , then we have  $q \succsim^* p$  which implies  $q \cdot x \geq p \cdot x$  contradicting  $q \cdot x < p \cdot x$ . On the other hand, if  $d(p, w) \cap H_{<}(q, w) \neq \emptyset$  and  $d(p, w) \cap H_{\leq}(q, w) \neq d(q, w) \cap H_{\leq}(p, w)$ , then it follows from the GWARP that  $d(q, w) \cap H_{\leq}(p, w) = \emptyset$ . This means that  $p \succ^* q$  and thus,  $q \cdot z \leq p \cdot z$ , contradicting  $q \cdot z > p \cdot z$ .  $\square$

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